[This question paper contains 4 printed pages.]

Sr. No. of Question Paper : 144 **E** **Your Roll No................**

Unique Paper Code : 235551

Name of the Course : **B.A. Programme**

Name of the Paper : Analysis

Semester : V

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. There are **3** sections.
3. Each section carries **25** marks.
4. Attempt any **two** parts from each question in each section.
5. Marks are indicated against each question.

SECTION I

1. (a) Define Supremum and Infemum of the set S. Find the Supremum and Infemum of the set :

$$S = \left\{1, 1+\frac{1}{2}, 1+\frac{1}{2}+\frac{1}{2^2}, \ldots\ldots, 1+\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^{n-1}}, \ldots\right\} \qquad (6)$$

(b) Comment on the following statements :

(i) Can a finite set be open ?

(ii) Can a finite non-empty set be open ?

(iii) Is every infinite set be open ?

Justify the answer by means of an example wherever needed. (6)

(c) Prove that if a function f is continuous on a closed and bounded interval [a, b], then it is bounded in [a, b]. (6)

2. (a) State Bolzano Weirstrass theorem. Verify the theorem for the set $S = \left\{\frac{1}{n}; n \in N\right\}$. (6.5)

(b) Show that the function f on [0, 1] defined as :

$f(x) = \frac{1}{3^n}$ when $\frac{1}{3^{n+1}} < x \le \frac{1}{3^n}$; (n = 0, 1, 2, 3....)

f(0) = 0

is discontinuous at $\frac{1}{3}, \frac{1}{3^2}, \frac{1}{3^3}, \ldots\ldots$ (6.5)

(c) Show that every uniformly continuous function on an interval is continuous on that interval, but the converse is not true. (6.5)

SECTION II

3. (a) Define convergent sequence. If $\langle a_n \rangle, \langle b_n \rangle$ be two sequences such that $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$, then show that $\lim_{n\to\infty}(a_n b_n) = ab$. (6.5)

(b) Show that $\lim_{n\to\infty}(n)^{1/n} = 1$ and hence show that

$$\lim_{n\to\infty} \frac{1}{n}\left(1 + 2^{1/2} + 3^{1/3} + \ldots + n^{1/n}\right) = 1$$ (6.5)

(c) Define monotonic sequence and show that $\langle a_n \rangle$ defined as $a_1 = 1$; $a_{n+1} = \frac{3+2a_n}{2+a_n}$; $n \ge 1$ is convergent and find its limit. (6.5)

4. (a) Let $\sum_{n=1}^{\infty} u_n$ is a positive term series such that

$$\lim_{n \to \infty} (u_n)^{1/n} = l.$$

Then prove that

(i) If $l < 1$ then $\sum_{n=1}^{\infty} u_n$ is convergent.

(ii) If $l > 1$ then $\sum_{n=1}^{\infty} u_n$ is divergent. (6)

(b) Test for convergence any **two** of the following series :

(i) $\sum_{n=1}^{\infty} \left[\frac{\sqrt{n+1} - \sqrt{n-1}}{n} \right]$

(ii) $\sum_{n=1}^{\infty} \frac{1.3.5. \ldots\ldots (2n-1)}{2.4.6 \ldots\ldots 2n} \cdot \frac{1}{2n+1}$

(iii) $\frac{1}{1.2} - \frac{1}{2.3} + \frac{1}{3.4} - \frac{1}{4.5} + \ldots\ldots$ (3+3)

(c) State Cauchy's General Principle for convergence of a series. Hence examine the convergence of the series $1 + \frac{1}{5} + \frac{1}{9} + \frac{1}{13} + \ldots\ldots + \frac{1}{4n-3} + \ldots\ldots$ (6)

SECTION III

5. (a) Let f be a monotonically increasing function defined on a closed and bounded interval [a, b]. Prove that f is Riemann integrable on [a, b]. (6)

(b) Discuss the convergence for any **two** of the following improper integrals :

(i) $\int_0^1 \frac{dx}{x^n}$ [By Definition]

(ii) $\int_0^{\pi} \frac{\sqrt{x}}{\sin x} dx$

(iii) $\int_0^{\infty} e^{-nx} x^m \, dx, \; m > 0, \; n > 0$ (3+3)

(c) Prove that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ and hence prove that $\int_0^1 x\sqrt{1-x^4} \, dx = \frac{\pi}{8}$. (6)

6. (a) Find Fourier series of function f, where $f(x) = |x|, -\pi \le x \le \pi$. (6.5)

(b) Test the pointwise and uniform convergence of $\langle f_n \rangle$ on $[0, 2\pi]$, where

$f_n(x) = \frac{\sin nx}{\sqrt{n}}, 0 \le x \le 2\pi, \; n \in N$. (6.5)

(c) Discuss the Riemann integrability of f over [0, 1], where

$$f(x) = (-1)^n, \; \frac{1}{n+1} < x < \frac{1}{n}, \; n \in N$$

and f(0) = 0, clearly stating the result used. (6.5)