

[This question paper contains 4 printed pages.]

Sr. No. of Question Paper : 143

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Your Roll No.....

Unique Paper Code : 235551

Name of the Course : B.A. Programme

Name of the Paper : Analysis

Semester : V

Duration : 3 Hours

Maximum Marks : 75

**Instructions for Candidates**

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. There are 3 sections.
3. Each section carries 25 marks.
4. Attempt any two parts from each question in each section.
5. Marks are indicated against each question.

**SECTION I**

1. (a) Give two examples of the following :

(i) A set which is bounded above but not bounded below.

(ii) A set which is bounded below but not bounded above.

(iii) A set which is neither bounded above nor bounded below. (6)

- (b) Define limit point of a set of real numbers. Find the limit points of the following sets :

$$B = \left\{ 1, 2, \frac{1}{2}, 3, \frac{1}{3}, 4, \frac{1}{4}, \dots \right\} \quad (6)$$

P.T.O.

- (c) Verify the continuity of the following function at  $x = 0$  :

$$f(x) = e^x \operatorname{sgn}(x + [x]), \quad x \neq 0$$

$$f(0) = 0.$$

where  $\operatorname{sgn}$  denotes the signum function and  $[x]$  is the greatest integer  $\leq x$ . (6)

2. (a) Show that the union of a finite collection of closed sets is a closed set. Give an example to show that the result may not be true for an infinite collection of closed sets. (6.5)

- (b) Show that the function  $f$  defined as

$$f(x) = \begin{cases} x & \text{if } x \text{ rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

is continuous only at  $x = 0$ . (6.5)

- (c) Prove that  $f(x) = x^2$  is not uniformly continuous on  $[0, \infty[$  whereas it is uniformly continuous on  $] -2, 2[$ . (6.5)

## SECTION II

3. (a) Show that the sequence  $\langle r^n \rangle$  converges to zero if  $|r| < 1$ . (6.5)

- (b) State Cauchy's General Principle of Convergence and show that the sequence

$$\left\langle 1 + \frac{1}{5} + \frac{1}{9} + \dots + \frac{1}{4n-3} \right\rangle \text{ is not convergent.} \quad (6.5)$$

- (c) Show that a necessary and sufficient condition for the convergence of a monotonic sequence is that it is bounded. (6.5)

4. (a) Prove that a necessary condition for the convergence of a series  $\sum_{n=1}^{\infty} u_n$  is  $\lim_{n \rightarrow \infty} u_n = 0$ . Justify with an example whether the condition is sufficient also for the convergence. (6)

(b) Test for convergence any two of the following series : (3+3)

$$(i) \frac{1}{3.7} + \frac{1}{4.9} + \frac{1}{5.11} + \frac{1}{6.13} + \dots$$

$$(ii) \sum_{n=1}^{\infty} \frac{\sin(n^2x + n)}{n^3 + n}, x \in \mathbb{R}$$

$$(iii) \sum_{n=1}^{\infty} \frac{1.3.5. \dots (2n-1)}{2.4.6. \dots 2n} \cdot \frac{1}{n}$$

(c) State Leibnitz test for the convergence of an alternating series

$$u_1 - u_2 + u_3 - u_4 + \dots, u_n > 0 \forall n.$$

Test for absolute convergence and convergence the series  $\frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$

(6)

### SECTION III

5. (a) Let  $f$  be a bounded function defined on a closed and bounded interval  $[a, b]$ . Define upper sum, lower sum, upper integral, lower integral of  $f$  and hence define Riemann integrability of  $f$  over  $[a, b]$ . Also show that a function  $f$  defined as

$$f(x) = 2 \forall x \in [0, 1] \text{ is Riemann integrable on } [0, 1]. \quad (6)$$

(b) Examine the convergence of any two of the following improper integrals :

$$(i) \int_0^1 \frac{\sqrt{x}}{\sin x} dx$$

$$(ii) \int_0^{\infty} e^{-x} x^2 dx$$

$$(iii) \int_1^2 \frac{dx}{(x-1)^3} \text{ [By Definition]} \quad (3+3)$$

P.T.O.

(c) Do any two parts :

(3+3)

(i) Show that  $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

(ii) Show that  $\int_0^1 \frac{x^2}{\sqrt{1-x^8}} dx = \frac{1}{8} B\left(\frac{3}{8}, \frac{1}{2}\right)$

(iii) Prove that a trigonometric series  $\sum_{n=1}^{\infty} \frac{\cos nx + \sin nx}{\sqrt{n}}$  is not a Fourier series.

6. (a) Find Fourier series of function  $f$ , where  $f(x) = x$ ,  $-\pi \leq x \leq \pi$ . (6.5)

(b) Show that the function  $f(x)$  defined on the interval  $[0, 4]$  as  $f(x) = x[x]$  is integrable on  $[0, 4]$ , stating clearly the results you are using. (6.5)

(c) Prove that  $\langle x^n \rangle$  is not uniformly convergent on  $[0, 1]$ , but is uniformly convergent on  $\left[0, \frac{1}{2}\right]$ . (6.5)