5228 F

Unique Paper Code	**:**	**235551**
Name of the Paper	**:**	**Analysis**
Name of the Course	**:**	**B.A. Programme - Mathematics**
Semester	**:**	**V**
Duration	**:**	**3 Hours**
Maximum Marks	**:**	**75 Marks**

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. There are **3** sections.
3. Each section carries **25** marks.
4. Attempt any **two** parts from each question in each section.
5. Marks are indicated against each question.

SECTION I

1. (a) Define a bounded set, its supremum and infemum. Find the supremum and infemum of the following sets:

(i) $A = \left\{-2, \frac{-3}{2}, \frac{-4}{3}, \ldots, \frac{-(n+1)}{n}, \ldots\right\}$

(ii) $B = \{-1, 2, -3, 4, -5, \ldots, (-1)^n n, \ldots\}$ (6)

(b) Prove that the intersection of a finite number of open sets is open. What happens if the family consists of infinite number of open sets? Justify the answer by means of an example. (6)

(c) Let $f(x)$ be the function defined on R by setting $f(x) = |x| + [x]$; for all $x \in R$. Determine the points of discontinuity of $f(x)$. $[x]$ is the greatest integer $\leq x$. (6)

2. (a) Define limit point of a set. Find the limit points of Z, the set of integers and Q, the set of rationals. (6.5)

(b) If a function f is continuous in $[a, b]$ and $f(a)f(b) < 0$, then show that there exists a point $c \in (a, b)$ such that $f(c) = 0$. (6.5)

(c) Show that the function $f(x) = \frac{1}{x^2}$ is uniformly continuous on $[a, \infty[,\ a > 0$. (6.5)

SECTION II

3. (a) Define convergent sequence. Show that every convergent sequence is bounded but the converse is not true. (6.5)

(b) Show that the sequence $\left\langle 1 + \frac{1}{4} + \frac{1}{7} + \ldots.. + \frac{1}{3n-2} \right\rangle$ is not convergent, while $\left\langle \frac{1}{n}\left(1 + \frac{1}{4} + \frac{1}{7} + \ldots.. + \frac{1}{3n-2}\right) \right\rangle$ is convergent. (6.5)

(c) Define monotonic sequence and show that $\langle S_n \rangle$, where $S_n = 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots\ldots + \frac{1}{n!}$ is convergent. (6.5)

4. (a) Prove that

$$\sum_{n=1}^{\infty} u_n \text{ is convergent } \Rightarrow \lim_{n\to\infty} u_n = 0.$$

Hence examine the convergence of the series $1 + \left(\frac{1}{2}\right)^{1/2} + \left(\frac{1}{3}\right)^{1/3} + \left(\frac{1}{4}\right)^{1/4} + \ldots\ldots$ (6)

(b) Test for convergence any **two** of the following series: (3+3)

(i) $\left(\frac{1}{3}\right) + \left(\frac{2}{5}\right)^2 + \left(\frac{3}{7}\right)^3 + \left(\frac{4}{9}\right)^4 + \ldots\ldots$

(ii) $\sum_{n=1}^{\infty} \frac{1.2.3. \ldots\ldots n}{7.10.13. \ldots\ldots (3n+4)}$

(iii) $\sum_{n=1}^{\infty} \left[\left(n^3+1\right)^{1/3} - n\right]$

(c) Define absolutely convergent series and conditionally convergent series. Also prove that the series $\frac{1}{\sqrt{1.2}} - \frac{1}{\sqrt{2.3}} + \frac{1}{\sqrt{3.4}} - \frac{1}{\sqrt{4.5}} + \ldots\ldots$ is conditionally convergent. (6)

SECTION III

5. (a) Show that a continuous function f defined on a closed and bounded interval $[a, b]$ is integrable. (6)

(b) Examine the improper integral $\int_a^b \frac{dx}{(x-a)^n}$ for convergence, using the definition of convergence.

OR

Test the convergence of the improper integral $\int_0^{\infty} e^{-x} x^{n-1}\, dx$. (6)

(c) Do any **two**: (3+3)

(i) Find the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{2^n}{n!} x^n$.

(ii) Prove that the trigonometric series $\cos x + \frac{\cos 2x}{\sqrt{2}} + \frac{\cos 3x}{\sqrt{3}} + \frac{\cos 4x}{\sqrt{4}} + \ldots\ldots$ is not a Fourier series.

(iii) Discuss the Riemann integrability of f on $[0, 2]$, where
$f(x) = [x]$, $[x]$ is the greatest integer $\leq x$.

6. (a) Find Fourier series of the function f, where

$$f(x) = 0, \quad -\pi \leq x \leq 0$$
$$f(x) = x, \quad 0 \leq x \leq \pi$$

(6.5)

(b) Prove that a sequence $\langle f_n \rangle$, where $f_n(x) = x^n$, is uniformly convergent on $\left[0, \frac{1}{2}\right]$ also prove that a series $\sum_{n=1}^{\infty} f_n$, where $f_n(x) = \frac{x^n}{n^2}$ is uniformly convergent on $[0, 1]$. (6.5)

(c) Prove that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ and hence prove that $\int_0^1 x\sqrt{1-x^4}\,dx = \frac{\pi}{8}$. (6.5)