

5227

F

Unique Paper Code : 235551

Name of the Paper : Analysis

Name of the Course : B.A. Programme - Mathematics

Semester : V

Duration : 3 Hours

Maximum Marks : 75 Marks

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. There are 3 sections.
3. Each section carries 25 marks.
4. Attempt any two parts from each question in each section.
5. Marks are indicated against each question.

SECTION I

1. (a) Define any two of the following illustrating each by means of an example:

- (i) Supremum of a set
- (ii) Limit point of a set
- (iii) Closed set.

(6)

- (b) Prove that the union of an arbitrary family of open sets is an open set. What can you say about the intersection of an arbitrary family of open sets? Justify the answer with example. (6)

- (c) Show that the function $f(x)$ defined on $[0, 1]$, as follows:

$$f(x) = \begin{cases} x + \frac{1}{2}; & 0 < x < \frac{1}{2} \\ 3x - \frac{1}{2}; & \frac{1}{2} < x < 1 \end{cases}$$

$$f(0) = 2; \quad f\left(\frac{1}{2}\right) = \frac{1}{2} \quad \text{and} \quad f(1) = 1$$

is discontinuous at $x = 0, \frac{1}{2}$, and 1. Also mention the type of discontinuity at each point. (6)

2. (a) State Bolzano Weirstrass theorem. Justify with the help of example that no condition can be relaxed in the theorem. (6.5)

- (b) If a function f is continuous on a closed and bounded interval $[a, b]$, then it attains its bounds in $[a, b]$. (6.5)

- (c) Show that $f(x) = \frac{1}{x}$ is not uniformly continuous on $]0, 1]$ whereas it is uniformly continuous on $[a, \infty[$, where $a > 0$. (6.5)

SECTION II

3. (a) State Cauchy's first theorem on limits and show that

$$\lim_{n \rightarrow \infty} \left[\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \dots + \frac{1}{\sqrt{n^2+n}} \right] = 1. \quad (6.5)$$

- (b) If $\langle a_n \rangle$, $\langle b_n \rangle$ and $\langle c_n \rangle$ be three sequences such that $a_n \leq b_n \leq c_n \quad \forall n \in N$ and $\lim_{n \rightarrow \infty} a_n = l = \lim_{n \rightarrow \infty} c_n$, then show that $\lim_{n \rightarrow \infty} b_n = l$. (6.5)

- (c) Define monotonic sequence and show that the sequence $\langle a_n \rangle$ defined by

$$a_1 = 5, \quad a_{n+1} = \sqrt{5 + a_n}, \quad n \geq 1$$

converges to the positive root of the equation $x^2 - x - 5 = 0$. (6.5)

4. (a) State Cauchy's convergence criterion for series. Hence examine the convergence of the

series $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots$ (6)

- (b) Test for convergence any **two** of the following series: (6)

(i) $\sum_{n=1}^{\infty} [\sqrt{n^4 + 2} - \sqrt{n^4 + 1}]$

(ii) $\frac{1}{2}x + \frac{2}{3}x^2 + \frac{3}{4}x^3 + \frac{4}{5}x^4 + \dots, \quad x > 0$

(iii) $\frac{1}{4.3} + \frac{1}{5.3^2} + \frac{1}{6.3^3} + \frac{1}{7.3^4} + \dots$

- (c) (i) Define sequence of partial sums of a series and find sequence of partial sums of the

series $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \dots$ (2)

(ii) Define an absolutely convergent series. Prove that the series $\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2}$ is absolutely

convergent. (4)

SECTION III

5. (a) Discuss the Riemann integrability of f defined as follows over $[0, 1]$, clearly stating the

results used: (4+2)

(i) $f(x) = \frac{1}{n}, \quad \frac{1}{n+1} < x \leq \frac{1}{n}, \quad n \in N$
 $f(0) = 0$

(ii) $f(x) = 2x^3 + 3x^2$ for $0 \leq x \leq 1$

(b) Examine the convergence of any **two** of the following improper integrals:

(3+3)

(i) $\int_0^{\pi/2} \frac{x^m}{(\sin x)^n} dx$

(ii) $\int_0^{\infty} \frac{x^2}{\sqrt{x^5+1}} dx$

(iii) $\int_0^1 \frac{dx}{x^3}$ [By Definition]

(c) Define Radius of convergence of a power series. Prove that a power series, having R as radius of convergence, converges absolutely and uniformly in $[-r, r]$, $0 < r < R$.

(6)

6. (a) Find Fourier series of function f defined as:

$$f(x) = -1, \quad -\pi \leq x < 0;$$

$$f(x) = 1, \quad 0 \leq x \leq \pi.$$

(6.5)

(b) Examine the uniform convergence of the sequence $\left\langle \frac{nx}{1+n^2x^2} \right\rangle$ on $[0, 1]$.

(6.5)

(c) Prove that $B(m, n) = 2 \int_0^{\pi/2} \sin^{2m-1} \theta \cos^{2n-1} \theta d\theta$, $m, n > 0$. Also state the relation between

Beta and Gamma function and prove that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

(6.5)