[This question paper contains 4 printed pages.]

Sr. No. of Question Paper : 238 **G** **Your Roll No...............**

Unique Paper Code : 235551

Name of the Paper : Analysis

Name of the Course : **B.A. Programme – Mathematics**

Semester : V

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on the receipt of this question paper.
2. There are **3** sections.
3. Each section consists of **25** marks.
4. Attempt any **two** parts from each question in each section.
5. Marks are indicated against each question.

SECTION I

1. (a) Define supremum and infimum of the set $S \subseteq R$. Find the supremum and infimum of the set

$$S = \left\{1, 1+\frac{1}{2}, 1+\frac{1}{2}+\frac{1}{2^2}, \ldots\ldots\ldots, 1+\frac{1}{2}+\frac{1}{2^2}+\ldots..+\frac{1}{2^{n-1}}, \ldots\ldots\right\} \quad (6)$$

(b) State the properties which make the set R of real umbers, a complete ordered field. (6)

(c) Define limit point of a set. Show that the set N of natural numbers has no limit point. (6)

2. (a) State Bolzano-Weierstrass theorem for sets. Prove that the set $\left\{3^n + \frac{1}{3^n};\ n \in N\right\}$ has no limit point. How does it contradict Bolzano-Weierstrass theorem ? (6.5)

(b) Test the continuity of the function

$$f(x) = \begin{cases} \frac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}}, & x \neq 0 \\ 0, & x = 0 \end{cases} \quad \text{at } x = 0$$ (6.5)

(c) Define uniform continuity of a function f on an interval I. Show that the function defined by $f(x) = x^3$ is uniformly continuous on $[-3, 3]$. (6.5)

3. (a) If $a_n \leq b_n \leq c_n$ for all n and $\langle a_n \rangle$ and $\langle c_n \rangle$ converge to l then $\langle b_n \rangle$ also converges to l. (6.5)

(b) State Monotone convergence theorem for sequence and hence prove that the sequence $\langle a_n \rangle$ defined as

$a_1 = 1,\ a_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots\ldots\ldots + \frac{1}{(n-1)!}\ (n \geq 2)$ is convergent. (6.5)

(c) State Cauchy's General Principle of convergence for sequence and show that the sequence $\langle a_n \rangle$ where $a_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots\ldots + \frac{1}{n}$ is not convergent. (6.5)

4. (a) Let $\sum_1^\infty u_n$ and $\sum_1^\infty v_n$ be two positive terms series such that $u_n \leq kv_n$ $\forall$ n, k being a fixed positive number then prove that

(i) $\sum u_n$ converges if $\sum v_n$ is convergent

(ii) $\sum v_n$ diverges if $\sum u_n$ is divergent (6)

(b) Test the convergence of the following series :

(i) $\frac{\sqrt{2}-\sqrt{1}}{1} - \frac{\sqrt{3}-\sqrt{2}}{2} + \frac{\sqrt{4}-\sqrt{3}}{3} - \ldots\ldots\ldots\ldots$

(ii) $\sum_{1}^{\infty}(-1)^n \frac{n+2}{2^n+5}$

(iii) $\sum_{1}^{\infty}(-1)^{n-1}\frac{1}{n}$ (6)

(c) State Cauchy's General Principle of convergence for an infinite series $\sum_{1}^{\infty} u_n$ and show that the series $\sum_{1}^{\infty}\frac{1}{n}$ does not converge. (6)

5. (a) A bounded function f is integrable on [a, b] iff for every $\varepsilon > 0$, there exists partition P of [a, b] such that $U(P, f) - L(P, f) < \varepsilon$. (6.5)

(b) Show that $\int_{-\infty}^{\infty} e^{-x^2}\,dx$ converges. (6.5)

(c) Show that $\int_0^2 x^4\left(8-x^2\right)^{-\frac{1}{2}}dx = \frac{16}{3}\beta\left(\frac{5}{3}, \frac{2}{3}\right)$. (6.5)

6. (a) Find the Fourier series of the function f where

$$f(x) = \begin{cases} -1, & \text{for } -\pi \le x < 0 \\ 1, & \text{for } 0 \le x \le \pi \end{cases}$$ (6)

(b) State Cauchy's uniform convergence criteria for a sequence of functions.

Test the sequence $\langle f_n(x) \rangle$ for uniform convergence, where

$$f_n(x) = \frac{\sin nx}{\sqrt{n}}, \quad 0 \le x \le 2\pi$$ (6)

(c) (i) Find the radius of convergence of the power series

$$\frac{1}{2}x + \frac{1.3}{2.5}x^2 + \frac{1.3.5}{2.5.8}x^3 + \ldots\ldots\ldots\ldots$$

(ii) If f is defined in [0, 1] by the condition

$$f(x) = (-1)^{r-1}, \text{ when } \frac{1}{r+1} < x \le \frac{1}{r}, \quad (r = 1, 2, 3, \ldots\ldots),$$

$f(0) = 0$. Show that $\int_0^1 f(x)\,dx = \log 4 - 1$ (6)