[This question paper contains 4 printed pages.]

4691 *Your Roll No.* ....

B.Sc. (G)/I **AS**

MATHEMATICS – Paper I

Algebra

(Admissions of 2004 and before)

Time : 3 Hours *Maximum Marks : 55*

(Write your Roll No. on the top immediately on receipt of this question paper.)

All *questions are compulsory.*
Attempt any **two** *parts from each question.*

1. (a) If '$\cdot$' is an associative binary composition in a non null set G, then show that $(G, \cdot)$ is a group if and only if

   (i) There exists an element $e \in G$ such that
   $$a \cdot e = a \quad \text{for all } a \in G; \text{ and}$$

   (ii) For each $a \in G$, there exists an element $b \in G$ such that
   $$a \cdot b = e$$ (4½)

   (b) Define a right coset of a subgroup H of a group G. Prove that there is one-one correspondence between any two right cosets of a subgroup H in G. (4½)

(c) Prove that the intersection of two normal subgroups of a group G is a normal subgroup of G. (4½)

2. (a) Prove that the only idempotent elements of an integral domain are 0 and 1. (4½)

(b) Define an ideal of ring and prove that the sum of two ideals of a ring is an ideal of the ring. (4½)

(c) Let R is a ring with unity. Prove that R is a commutative ring if and only if

$(ab)^2 = a^2 b^2 \; ; \; \forall \; a, b \in R$ (4½)

3. (a) Let C be a field of complex numbers. Prove that C is a vector space over C. (4½)

(b) Define Linear Combination of vectors $\alpha_1, \alpha_2, ---, \alpha_n \in V$ where V is a vector-space over field F. Find the value of K for which the vector $x = (1, K, 5)$ is a linear combination of $x_1 = (1, -3, 2)$ and $x_2 = (2, -1, 1)$. (4½)

(c) If V is n-dimensional vector-space, prove that every set of n vectors which is linearly independent is a basis of V. (4½)

4. (a) Solve the equation

$4x^4 - 24x^3 + 31x^2 + 6x - 8 = 0$; given that the sum of two roots is zero. (4½)

(b) If $\alpha + \beta + \gamma = 0$, prove that

$$3(\alpha^2+\beta^2+\gamma^2)(\alpha^5+\beta^5+\gamma^5) = 5(\alpha^3+\beta^3+\gamma^3)(\alpha^4+\beta^4+\gamma^4)$$

(4½)

(c) Find the equation whose roots are the roots of the equation

$x^3 - 3x^2 + 2x - 5 = 0$, each diminished by 2. (4½)

5. (a) Prove that

$$[(\cos\alpha + i\sin\alpha) - (\cos\beta + i\sin\beta)]^8 + [(\cos\alpha - i\sin\alpha) - (\cos\beta - i\sin\beta)]^8 = 2^9 \sin^8 \tfrac{1}{2}(\alpha - \beta)\cos^4(\alpha + \beta).$$

(4½)

(b) Find an expression for $\cos^7\theta \sin^5\theta$ in terms of Sines of multiples of θ. (4½)

(c) Solve the equation

$$Z^6 - Z^5 + Z^4 - Z^3 + Z^2 - Z + 1 = 0.$$ (4½)

6. (a) Find the rnk of Matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & -2 & 1 \\ 2 & 0 & -3 & 2 \\ 3 & 3 & 0 & 3 \end{pmatrix}$$

by reducg it into Normal form. (5)

(b) For what values of λ does the following system of equation have a solution

$$x + y + z = 1$$

$$x + 2y + 4z = \lambda$$

$$x + 4y + 10z = \lambda^2$$

and also find the solution in each case. (5)

(c) State Cayley-Hamilton-Theorem. Find the characteristic equation of the matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 0 & -1 \end{bmatrix}$$

and hence calculate its cube. (5)