[This question paper contains 4 printed pages.]

5007 *Your Roll No............*

B.Sc. (G) / I **B**

MATHEMATICS – Paper I

(Algebra)

Time : 3 Hours *Maximum Marks : 55*

(Write your Roll No. on the top immediately on receipt of this question paper.)

All *questions are compulsory.*
Attempt any **two** *parts from each question.*
The **first** *question carries* **ten** *marks and rest of the questions carry* **nine** *marks each.*

1. (a) Prove that a subgroup N of a group G is a normal subgroup if and only if every left coset of N in G is a right coset of N in G and vice-versa. Hence show that every subgroup of index 2 is normal in G. (2.5+2.5)

   (b) Define order of an element in a group. Show that if in a group G, $0(a) = 2 \quad \forall\ a \in G,\ a \neq e$ then G is a commutative group. (1+4)

   (c) Prove that every group of order less than 6 is commutative. (5)

2. (a) Define a ring. Show that if $(G, *)$ is a commutative group, then $< G, +, \cdot >$ is a commutative ring with respect to

Ring addition : $a + b = a * b \quad \forall\ a, b \in G$

Ring multiplication : $a . b = e \quad \forall\ a, b \in G$

where e is the identity element in G. (1.5+3)

(b) Show that a commutative ring R is an integral domain if and only if for $a, b, c \in R$ with $a \neq 0$, $ab = ac \Rightarrow b = c$. (4.5)

(c) Prove that the intersection of two right ideals is a right ideal. What about the intersection of a right ideal with a left ideal ? Justify your answer. (2+2.5)

3. (a) Every field is a vector space over itself. Verify this fact for the field $< \mathbb{R}, +, \cdot >$ of real nos. That is show that $(\mathbb{R}, +)$ is a vector space over $< \mathbb{R}, +, \cdot >$. (4.5)

(b) Prove that the intersection of two subspaces of a vector space V(F) is a subspace of V(F). Show by an example that $W_1 \cup W_2$ for two subspaces W_1 and W_2 of V(F), may not be a subspace of V(F). (2.5+2)

(c) Define basis of a vector space. Show that the vectors (1,2,1), (1,0,–1) and (0,–3,2) form a basis of $\mathbb{R}^3(\mathbb{R})$. (1+3.5)

4. (a) Define Hermitian, skew Hermitian matrices. If A and B are square matrices of same order, show that $B^{\theta}AB$ is Hermitian or skew-Hermitian according as the matrix A is Hermitian or skew-Hermitian. (1+3.5)

(b) Show that the only real value of λ for which the following system of linear equations has a non-zero solution is 6 and then solve the equations :

$$x + 2y + 3z = \lambda x$$
$$3x + y + 2z = \lambda y$$
$$2x + 3y + z = \lambda z$$
(4.5)

(c) Prove that the characteristic roots of a triangular matrix are its principal-diagonal elements. Use it to find the characteristic roots of

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

Also find the characteristic vectors of A.

(2+.5+2)

5. (a) If $x_r = \cos\frac{\pi}{2^r} + i\sin\frac{\pi}{2^r}$, use De-Moivre's Theorem to prove that :

$$x_1 x_2 x_3 \text{-----} \text{ infinity} = -1.$$
(4.5)

(b) Find the sum of the infinite series :

$$\cos\theta - \frac{1}{2}\cos 2\theta + \frac{1}{3}\cos 3\theta - \text{---} \text{ ad. inf}$$
(4.5)

(c) State De-Moivre theorem for fractional indices. Use it to solve the equation :

$$x^7 - x^4 + x^3 - 1 = 0 \qquad (1.5+3)$$

6. (a) The roots of the equation $3x^3 - x^2 - 3x + 1 = 0$ are in H.P. find them. (4.5)

(b) If α, β, γ are the roots of the equation $x^3 + qx + r = 0$, $\gamma \neq q$, find the values of

(i) $\sum (\beta + \gamma - 2\alpha)$

(ii) $\sum \alpha^3 \beta^3$ and

(iii) $\sum \frac{1}{\beta + \gamma}$ (1.5×3)

(c) If α, β, γ are the roots of the equation :

$$x^3 - px^2 + qx - r = 0, \; r \neq 0$$

find the equation whose roots are

$$\beta\gamma + \frac{1}{\alpha}, \; \gamma\alpha + \frac{1}{\beta}, \; \alpha\beta + \frac{1}{\gamma} \qquad (4.5)$$