[This question paper contains 4 printed pages.]

107 *Your Roll No.* ....

B.Sc. (G) / I **C**

MATHEMATICS – Paper I

(Algebra)

Time : 3 Hours *Maximum Marks : 55*

(Write your Roll No. on the top immediately on receipt of this question paper.)

All *questions are compulsory.*
Attempt any **two** *parts from each questions.*
Each of the first **five** *questions carry* **nine** *marks and the last question carries* **ten** *marks.*

1. (a) Define an abelian group and show that the set I of all integers is an abelian group under the binary operation * defined by

$$a * b = a + b + 1 \quad \text{for all } a, b \in I$$

(b) Define a subgroup. Prove that a non-empty finite subset H of a group G is a subgroup of G if and only if H is closed for the group operation of G.

(c) Prove that if H is a subgroup of index 2 in a group G, then H is a normal subgroup of G.

P.T.O.

2. (a) Let M be the set of all 2×2 matrices over integers under matrix addition and matrix multiplication. Prove that M is a ring. Is it commutative ring? Justify your answer.

(b) Prove that the necessary and sufficient condition for a non-empty subset S to be a subring of a ring $(R, +, \cdot)$ are

(i) $a, b \in S \Rightarrow a - b \in S$; and

(ii) $a, b \in S \Rightarrow a \cdot b \in S$

(c) Prove that a commutative ring D is an integral domain if and only if $a, b, c \in D$ with $a \neq 0$, the relation $ab = ac$ implies $b = c$.

3. (a) Define a subspace of a Vector space. Show that the set

$W = \{(a_1, a_2, a_3) : a_1 + a_2 + a_3 = 0, \ a_1, a_2, a_3 \in R\}$

is a subspace of $R^{(3)}(R)$.

(b) If S is non-empty subset of vector space V over F, then prove that L(S), linear-span of S, is a subspace of V.

(c) Prove that the vectors (1, –3, 2), (2, 4, 1) and (1, 1, 1) form a basis of $R^{(3)}(R)$.

4. (a) Solve the equation :

$$x^4 - 2x^3 + 4x^2 + 6x - 21 = 0,$$

being given that it has two roots equal in magnitude but opposite in sign.

(b) Find the sum of the reciprocals of the fifth powers of α, β, γ - the roots of the equation

$$x^3 + 2x^2 + 1 = 0$$

(c) If α, β, γ be the roots of the equation $x^3 - px^2 + qx - r = 0,\ r \neq 0$, form the equation whose roots are $\beta\gamma + \frac{1}{\alpha},\ \gamma\alpha + \frac{1}{\beta},\ \alpha\beta + \frac{1}{\gamma}$.

5. (a) Prove that

$$\left(\frac{1 + \sin\theta + i\cos\theta}{1 + \sin\theta - i\cos\theta}\right)^n = \cos n\left(\frac{\pi}{2} - \theta\right) + i\sin n\left(\frac{\pi}{2} - \theta\right),$$

n being an integer, provided $\theta \neq 2K\pi - \frac{\pi}{2}$.

(b) Sum the series

$\cos\theta + x\cos 2\theta + x^2\cos 3\theta + \ldots$.. to n terms.

(c) Solve the equation

$(Z+1)^{2n} + (Z-1)^{2n} = 0$ and prove that the roots are purely imaginary.

6. (a) Define rank of the matrix and find rank of the matrix :

$$A = \begin{pmatrix} 1 & 2 & -4 & 5 \\ 2 & -1 & 3 & 6 \\ 8 & 1 & 9 & 7 \end{pmatrix}$$

(b) For what values of λ and μ do the following system of equations :

$$x + y + z = 6$$
$$x + 2y + 3z = 10$$
$$x + 2y + \lambda z = \mu$$

have (i) a unique solution (ii) No solution (iii) an infinite number of solution.

(c) State Cayley Hamilton Theorem and using it, find the inverse of the matrix :

$$\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$