This question paper contains 3 printed pages.]

Your Roll No. ....

1475

B.A./B.Sc. (Hons.)/III A

MATHEMATICS – Paper XVI

(Analysis-V)

Time : 2 Hours **Maximum Marks : 38**

(Write your Roll No. on the top immediately on receipt of this question paper.)

Attempt any **two** parts from each question.

Section – I

1. (a) If $<V_n>$ is a positive monotonically decreasing sequence with limit zero and if the sequence of partial sums for an infinite series $\sum_{n=1}^{\infty} U_n$ is bounded then prove that the series $\sum_{n=1}^{\infty} U_n V_n$ converges.

Also show that the series

$$1+\frac{1}{2}-\frac{1}{3}-\frac{1}{4}+\frac{1}{5}+\frac{1}{6}-\frac{1}{7}-\frac{1}{8}+ \ldots\ldots\ldots$$

converges. 4 + 1

(b) For $|x| < 1$, convert the series

$$\frac{x}{1+x} - \frac{2x^2}{1+x^2} + \frac{3x^3}{1+x^3} - \dots\dots\dots\dots$$

into a double series and prove that it is equal to

$$\frac{x}{(1+x)^2} - \frac{x^2}{(1+x^2)^2} + \frac{x^3}{(1+x^3)^2} - \dots\dots\dots\dots \quad 5$$

(c) Prove that for $|x| < 1$

$$\frac{1}{2}\{\log(1-x)\}^2 = \sum_{n=1}^{\infty}\left(1 + \frac{1}{2} + \dots + \frac{1}{n}\right)\frac{x^{n+1}}{n+1} \quad 5$$

Section – II

2. (a) State and prove Cauchy's general principle of uniform convergence for a sequence of functions defined on an interval [a, b]. 5

(b) (i) Show that the sequence $<f_n>$ where

$$f_n(x) = \frac{\log(1+n^3x^2)}{n^2} \; \forall n, \forall x \in [0, 1]$$

converges uniformly on [0, 1]. 3

(ii) Show that the series $\sum_{n=1}^{\infty} \frac{x^n}{n(n+1)}$ converges uniformly on [–1, 1]. 2

(c) (i) If a sequence of functions $<f_n>$ converges uniformly to f on [a, b] and if each f_n is continuous on [a, b] then prove that f is continuous on [a, b]. 3

(ii) Examine the sequence $<f_n>$ where $f_n(x) = x^n \; \forall n, \forall x \in [0, 1]$.

For uniform convergence on [0, 1]. 2

Section – III

3. (a) Let f be a bounded and integrable function on $[-\pi, \pi]$. If a_n's and b_n's are Fourier coefficients of f, then prove that **5**

$$\sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right) \leq \frac{1}{\pi} \int_{-\pi}^{\pi} \{f(x)\}^2 \, dx$$

(b) Find the Fourier series of the periodic function f with period 2π defined as

$$f(x) = \begin{cases} x & \text{when } -\pi < x \leq 0 \\ 2x & \text{when } 0 < x \leq \pi \end{cases}$$

Deduce that

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\ldots$$ **4 + 1**

(c) Find the Fourier series which represents the function $|\cos x|$ in the interval $[-\pi, \pi]$. **5**

Section – IV

4. (a) If a power series $\sum_{n=0}^{\infty} a_n x^n$ has interval of convergence $]-R, R[$ and converges at $x = R$ then prove that it converges uniformly on $[0, R]$. **4**

(b) Prove that

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\ldots\ldots, -1 < x \leq 1$$ **4**

(c) Define exponential function $E(x)$ as the sum of a power series and prove that :

(i) $E(x + y) = E(x)\,E(y)\ \forall\, x, y \in \mathbb{R}$

(ii) $E(x) = e^x\ \forall x \in \mathbb{R}$ where $e = E(1)$ **1 + 1 + 2**
