This question paper contains 7 printed pages.]

Your Roll No. ....

1474

B.A./B.Sc. (Hons.)/III

MATHEMATICS – Unit XV

(Analysis – IV)

Time : 2 Hours **Maximum Marks : 38**

(Write your Roll No. on the top immediately on receipt of this question paper.)

Attempt **all** questions.

Section – I

1. Let f be a bounded function defined on [a, b]. Suppose f is continuous on [a, b] except at point $c \in (a, b)$. Show that f is Riemann integrable on [a, b].

OR

Let f be a bounded and monotonically decreasing function defined on [a, b]. Show that f is Riemann integrable on [a, b]. 3

2. If f is Reimann integrable on [a, b] and has a primitive G on [a, b], show that

$$\int_a^b f = G(b) - G(a)$$

OR

Suppose f is continuous on [a, b] and g is Riemann integrable on [a, b] and $g(x) \geq 0 \ \forall \ x \in [a, b]$. Show that there exists a $c \in [a, b]$, such that

$$\int_a^b f(x)\, g(x)\, dx = f(c) \int_a^b g(x)\, dx$$ 4

3. Using definition of Riemann-Stieltjes integral, evaluate one of the following integrals.

$$\int_0^2 x^3\, d[x]$$

OR

$$\int_0^2 x^2 \, d(x - [x])$$

where $[x]$ denotes the greatest integer $\leq x$. **3**

Section – II

4. Show that $\int_0^\infty \frac{\cos x}{\sqrt{x^2 + x}} dx$ converges.

OR

Show that $\int_0^\infty \frac{x \log x}{(1 + x^2)^2} dx$ converges. **3**

5. Evaluate the integral

$$\int_0^1 x^4 \left(\log\left(\frac{1}{x}\right)\right)^3 dx$$

OR

Show that

$$\int_0^1 \frac{x^2}{\sqrt{1 - x^4}} dx \times \int_0^1 \frac{dx}{\sqrt{1 + x^4}} = \frac{\pi}{\sqrt[4]{2}}$$ **3**

6. Show that

$$\int_0^1 x^{\frac{-1}{3}}(1-x)^{\frac{-2}{3}}(1+2x)^{-1}$$

$$= \frac{1}{9^{\frac{1}{3}}}\beta\left(\frac{2}{3}, \frac{1}{3}\right)$$

OR

Show that, if p, q > 0, then

$$\int_{-1}^{1} (1+x)^{p-1}(1-x)^{q-1}\,dx$$

$$= 2^{p+q-1}\,\beta(p, q).$$ 3

Section – III

7. Evaluate one of the following integrals by using differentiation under the integral sign.

$$\int_0^\infty \frac{\tan^{-1}(ax)}{x(1+x^2)}\,dx,\ a \geq 0.$$

OR

$$\int_0^{\pi/2} \frac{\log(1+y\sin^2 x)}{\sin^2 x}\,dx,\ y > 0.$$ 3

8. State Green's theorem and use it to evaluate the line integral

$$\oint_C (x + y^2)dx + (x^2 - y)dy$$

where C is the closed curve formed by $y^3 = x^2$ and $y = x$ between (0, 0) and (1, 1).

OR

Use Green's theorem to evaluate the line integral

$$\oint_C (xy + x + y)dx + (xy + x - y)dy$$

where C is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$ 3

9. Evaluate the double integral

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \frac{dy\,dx}{(1 + e^y)\sqrt{1 - x^2 - y^2}}$$

OR

Show that

$$\int_0^1 \int_x^{1/x} \frac{y\,dy\,dx}{(1 + xy)^2 (1 + y^2)} = \frac{1}{4}(\pi - 1)$$ 4

Section – IV

10. Evaluate the triple integral

$$\iiint_E \frac{dx\,dy\,dz}{(x+y+z+1)^3}$$

where E is the interior of the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$, $x + y + z = 1$.

OR

Show that

$$\iiint_E \frac{dx\,dy\,dz}{\sqrt{x^2+y^2+(z-1)^2}} = \frac{\pi}{6}$$

where E is given by

$4(x^2 + y^2 + z^2) \leq 1$ **4**

11. Verify the Gauss divergence theorem for

$\vec{F} = (2x - z)\hat{i} + x^2y\hat{j} - xz\hat{k}$

taken over the region bounded by

$x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$

OR

Show that

$$\iiint z \, dx \, dy \, dz = \frac{\pi}{4} h^4 \cot\theta \cot\phi$$

taken throughout the volume bounded by the cone

$$z^2 = x^2 \tan^2\theta + y^2 \tan^2\phi$$

and the planes $z = 0$ and $z = h$. 5
