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Your Roll No...............

1471

B.A./B.Sc. (Hons.)/III **A**

MATHEMATICS—Unit 12

(Algebra–III)

Time : 2 Hours *Maximum Marks : 38*

(Write your Roll No. on the top immediately on receipt of this question paper.)

Attempt *one* question from each Section.

Section I

1. (*a*) Show that a division ring has no zero divisors. 2

(*b*) Prove that if A and B are two ideals of a ring R, then $A + B = \langle A \cup B \rangle$, where, for any subset $S \subseteq R$, $\langle S \rangle$ denotes the ideal of R generated by S. 3

(*c*) Prove that the characteristic of a finite non-zero integral domain must be a prime number. Give example of an infinite integral domain with finite characteristic. 4½

P.T.O.

2. (*a*) If U is an ideal of a ring R and $r(U) = \{x \in R \mid xu = 0 \ \forall u \in U\}$, prove that $r(U)$ is an ideal of R. 2½

(*b*) If R is a division ring, then show that the centre Z(R) of R is a field. 2½

(*c*) If A is a left ideal and B is a right ideal of a ring R, prove that :

(*i*) AB is an ideal of R

(*ii*) BA need not be even a one-sided ideal of R. 4½

Section II

3. (*a*) Prove that any ring with unity can be imbedded into a ring of endomorphisms of some additive abelian group. 5½

(*b*) Let I be an ideal of a ring R such that $I \neq R$. For an element $a \in R$, $a \notin I$, if $I + (a) = R$, prove that I is a maximal ideal of R and conversely. 4

4. (*a*) In the ring of integers, prove that an ideal is maximal if and only if it is generated by a prime number. 4½

(*b*) If D_1 and D_2 are two isomorphic integral domains, prove that their respective field of quotients F_1 and F_2 are also isomorphic.

Is the converse true ? Justify. 5

Section III

5. (*a*) Prove that the ring $Z[i]$ of Gaussian integers is a Euclidean domain. 4½

(*b*) If R is a commutative ring with unity such that $R[x]$ is a principal Ideal Domain, then prove that R is a field. Hence or otherwise prove that $Z[x]$ is not a principal Ideal Domain. 5

6. (*a*) Prove that an element 'a' in a Euclidean domain R is a unit if and only if $\delta(a) = \delta(1)$, where δ is a Euclidean valuation on R. 3½

(*b*) Let R be a commutative ring with unity and A an ideal of R. Show that : 6

(*i*) $\frac{R[x]}{A[x]} \cong \frac{R}{A}[x]$.

(*ii*) Hence or otherwise prove that if A is a prime ideal of R then A[x] is a prime ideal of R[x].

Section IV

7. (*a*) Find the degree of the splitting field of the polynomial $x^4 + x^2 + 1$ over the field Q of rational numbers.

Find a basis of this splitting field over Q. 5

(*b*) Prove that it is impossible to construct a regular septagon using ruler and compass alone. 4½

8. (*a*) Let L be an algebraic extension of a field K and K be an algebraic extension of a field F.

Prove that L is an algebraic extension of F. 5½

(*b*) Find the degree of the splitting field of the polynomial $x^6 + 1$, over the field Q of rational numbers. 4