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Your Roll No......................................

1472

B.A./B.Sc. (Hons.)/III A

MATHEMATICS—Unit XIII

(Algebra—IV)

Time : 2 *Hours* *Maximum Marks* : 38

(Write your Roll No. on the top immediately on receipt of this question paper.)

Attempt any *two* questions *from each Section.*

Section I

1. Define a finite-dimensional vector space and show that any two bases of a finite-dimensional vector space have same number of elements. 4½

2. Define L(S), linear span of any subset S of a vector space V(F).

   (*a*) Check whether (3, –4, 6) in the subspace of R^3 (R) is spanned by vectors (1, 2, –1), (2, 2, 1) and (1, –2, 3) ? 2½ (*b*) Find the two subspaces A and B of R^4 (R) such that

$\dim A = 2, \dim B = 3, \dim (A \cap B) = 1.$ 2

3. Show that the union of *two* :

(*a*) Subspaces of a vector-space may not be a subspace. 2

(*b*) Let $V = R^3$ (R) and W be the subspace generated by $\{(1, 2, 3)\}$. Find a subspace W_1 of V such that $V = W_1 \oplus W$. 2½

Section II

4. (*a*) State and prove Sylvester's law for rank and nullity of a linear transformation from V to W, where V is a finite-dimensional vector space. 3

(*b*) Let T be a linear operator on a finite-dimensional vector space V and let Rank T^2 = Rank T. Show that Range $T \cap \ker T = \{0\}$. 2

5. Define mapping $T : R^3 \to R^3$ as

$T(x_1, x_2, x_3) = (3x_1 + x_3, -2x_1 + x_2, -x_1 + 2x_2 + 4x_3).$

(*i*) Show that T is a linear operator on R^3. 5

(*ii*) Prove that T is invertible and also give a rule for T^{-1} like the one which defines T.

6. Let $T : V \rightarrow V$ be a linear transformation (V is a vector space over the field F) and let β and β' be two ordered basis of V. Then prove that $[T]_{\beta'}$ and $[T]_{\beta}$ are similar. 5

Section III

7. If W is a subspace of a finite-dimensional vector space V (F), then show that dim A (W) = dim V – dim W. Also deduce that : 4½

$$\frac{\widehat{V}}{A(W)} \cong \widehat{W}$$

where $\widehat{V}$ denotes dual of V.

8. (*a*) Let V be a finite-dimensional vector space over the field F and $v_1 \neq v_2$ are in V, then prove that there exists a $f \in \widehat{V}$ such that $f(v_1) \neq f(v_2)$. 2

(*b*) Given $\alpha_1 = (1, 3)$ and $\alpha_2 = (2, 1)$ in R^2 (R). Find $\alpha \in R^2$ (R) such that

$$< \alpha, \alpha_1 > = 3 \text{ and } < \alpha, \alpha_2 > = -1,$$

where < , > denotes the standard inner product in R^2. 2½

9. Let V be the set of real-valued functions $y = f(x)$ satisfying

$$\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0.$$

Prove that V is a 3-dimensional real vector space and in V define inner product $(f, g) = \int_{-\infty}^{0} fg\, dx$.

Find an orthonormal basis of V over R. 4½

Section IV

10. Find the eigenvalues and bases of corresponding eigen spaces of the matrix. 5

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{bmatrix}.$$

Is A similar to a diagonal matrix ? Justify.

11. Let T be a linear opeartor on R^2, the matrix of which in the standard ordered basis is $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

(*a*) Prove that the only subspaces of R^2 invariant under T are R^2 and zero subspaces. 3

(*b*) If a diagonalizable operator has the only charateristic values 0 and 1, then is it a projection ? 2

12. Let V (F) be a vector space and $W_1, W_2, \ldots\ldots, W_k$ be subspaces of V, then prove that :

$V = W_1 \oplus W_2 \oplus \ldots\ldots \oplus W_k$ if and only if there exist k linear operators $E_1, E_2, \ldots\ldots, E_k$ on V, such that : 5

(*i*) Each E_i is a projection.

(*ii*) $E_i E_j = 0$ for all $i \neq j$.

(*iii*) $I = E_1 + E_2 + \ldots\ldots + E_k$

(*iv*) The range of $E_i = W_i$ for all i.