[This question paper contains 3 printed pages.]

1476 *Your Roll No.* ....

B.A./B.Sc. (Hons.)/III **A**

MATHEMATICS–Paper XVII & XVIII (*i*)

(Number Theory)

Time : 2 Hours *Maximum Marks : 38*

(Write your Roll No. on the top immediately on receipt of this question paper.)

Attempt any **two** *parts from each question. Marks are indicated against each question.*

1. (a) Obtain three consecutive integers, the first of which is divisible by a square, the second by a cube and the third by a fourth power. (4)

(b) (i) If p and q are distinct primes, prove that $p^{q-1} + q^{p-1} \equiv 1 \pmod{pq}$.

(ii) Use Fermat's Theorem to prove that if p is an odd prime, then

$$1^{p-1} + 2^{p-1} + \cdots + (p-1)^{p-1} \equiv -1 \pmod{p}$$

(2,2)

(c) Show that $(m-1)! \equiv -1 \pmod{m}$ holds if and only if m is prime. (4)

P.T.O.

2\. (a) Define Euler ϕ-function. Show that

$$\sum_{d/n} \phi(d) = n$$

verify the above result for $n = 30$. (4,1)

(b) If n is a positive integer and p a prime, then show that exponent of the highest power of p that divides n! is $\sum_{k=1}^{\infty}\left[\frac{n}{p^k}\right]$. Hence find the number of zeros with which the decimal representation of 50! terminates. (3,2)

(c) Prove that

$$\pi d = n^{\tau(n)/z}$$
$$d/n$$

where $\tau(n)$ denotes the number of positive divisors of n. (5)

3\. (a) Let gcd $(a,n) = 1$ and let $a_1, a_2, \text{---} a_{\phi(n)}$ be the positive integers less than n and relatively prime to n. If a is primitive root of n, then show that $a, a^2, \text{---} a^{\phi(n)}$ are congruent modulo n to $a_1, a_2, \text{---} a_{\phi(n)}$ in some order. Hence show that if n has a primitive root, then it has exactly $\phi(\phi(n))$ of them. (3,2)

(b) Let p be as odd prime and gcd $(a, p) = 1$ then show that a is quadratic residue of p if and only if

$$a^{\frac{p-1}{2}} \equiv 1 \pmod{p}$$ (5)

(c) Let p be an odd prime and let gcd (a, p) = 1. If n denotes the number of integers in the set

$$S = \left\{a, 2a, 3a, ---- \frac{1}{2}(p-1)a\right\}$$

whose remainders upon division by p exceed p/2. Then show that

$$(a/p) = (-1)^n \qquad (5)$$

4. (a) Define Perfect number. Show that an even perfect number n ends in digit 6 or 8. (1,4)

(b) Define Mersenne Numbers M_n $(n \geq 1)$. Show that if p and q = 2p + 1 are primes, then either q/M_p or $q/(M_p + 2)$ but not both. Further, show that M_{23} is a composite number. (3,2)

(c) Show that any prime p can be written as sum of four squares. (5)