[This question paper contains 4 printed pages.]

2511 *Your Roll No.* ....

B.Sc. (G)/III **A**

MATHEMATICS – Paper V

(Real Analysis)

Time : 3 Hours *Maximum Marks : 55*

(Write your Roll No. on the top immediately on receipt of this question paper.)

Attempt any **two** *parts from each question. Question No.* **3** *carries* **10** *marks and others carry* **9** *marks each.*

1. (a) State the properties which make the set of real numbers $\mathbb{R}$ a complete ordered field.

   (b) Define an open set. Prove that the union of an arbitrary family of open sets is an open set. Is an intersection of arbitrary family of open sets is always an open set ? Justify your answer.

   (c) Define limit point of a set. Give an example of a set having two limit points. Show that every real number is a limit point of the set Q of rational numbers.

2. (a) Show that the sequence $\langle a_n \rangle$ defined by $a_n = \gamma^n$ converges to zero if $|\gamma| < 1$.

(b) Show that the sequence $<a_n>$ defined by

$$a_n = 1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n-1}$$ does not converge.

(c) State Monotone Convergence Theorem. Show that the sequence $<a_n>$, where

$$a_1 = 1, \quad a_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{(n-1)!}, n \geq 2$$

converges.

3. (a) State and prove Cauchy's n^{th} root test.

(b) Test for convergence of the following series.

(i) $1 + \frac{x^2}{2} + \frac{x^3}{3} + \ldots.$ for $x > 0$

(ii) $\frac{1}{2}x + \frac{1.3}{2.4}x^2 + \frac{1.3.5}{2.4.6}x^3 + \ldots.$

for all +ve values of x.

(c) Define conditional convergence of an infinite series. Give an example of a conditionally convergent series. Show that the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{\sin n\alpha}{n^3}, \ \alpha \text{ being real,}$$

is absolutely convergent.

4. (a) Let f be the function defined on [−1, 1] by :

$$f(x) = \begin{cases} x, & \text{if } x \text{ is irrational,} \\ 0, & \text{if } x \text{ is rational.} \end{cases}$$

Show that f is continuous only at $x = 0$.

(b) Show that every function defined and continuous on a closed and bounded interval attains its suprimum.

(c) Define uniform continuity of a function. Show that the function :

$$f(x) = \sin\frac{1}{x}, \text{ for all } x \in]0, \infty[$$

is not uniformly continuous.

5. (a) State and prove Role's Mean Value theorem.

(b) Prove that between any two real roots of $e^x \sin x = 1$, there is at least one real root of $e^x \cos x + 1 = 0$.

(c) Obtain Maclaurin's Series expansion of

$f(x) = \log(1 + x), -1 < x \leq 1.$

6. (a) Prove that $\frac{x}{1 + x^2} < \tan^{-1} x < x$, if $x > 0$.