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*Your Roll No.* ....................................

**8061**

**B.Sc. (G)/III/NS** **D**

MATHEMATICS—Paper V

(Real Analysis)

*Time* : 3 *Hours* *Maximum Marks* : 55

*(Write your Roll No. on the top immediately on receipt of this question paper.)*

Attempt any *two* parts from each question.

*Third* question carries *ten* marks and

others carry *nine* marks each.

1. (*a*) Let $x$ and $y$ be any two real numbers, then show that :

$$|x - y| \geq ||x| - |y||,$$

and the equality holds if $xy \geq 0$.

(*b*) Prove that the intersection of two open sets is open. Is the conclusion true for an arbitrary family of open sets ? Justify your answer.

(*c*) Show that the derived set of a bounded set is bounded. $4\frac{1}{2}, 4\frac{1}{2}, 4\frac{1}{2}$

2. (*a*) Define a Cauchy sequence. Show that every convergent sequence is a Cauchy sequence.

(*b*) Show that the sequence $\langle a_n \rangle$, where :

$$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots\dots\dots + \frac{1}{n}$$

cannot converge.

(*c*) Show that the sequence $\langle a_n \rangle$, define by :

$$a_1 = 1; \; a_{n+1} = \frac{4 + 3a_n}{3 + 2a_n}, \; n \geq 1$$

converges. What is limit of $\langle a_n \rangle$ ? $4\frac{1}{2}, 4\frac{1}{2}, 4\frac{1}{2}$

3. (a) If a positive terms series $\sum_{n=1}^{\infty} u_n$ is convergent, then show that $\lim_{n \to \infty} u_n = 0$. Is the converse true ? Justify your answer with an example.

(b) State and prove D'Alembert's Ratio Test.

(c) Test for convergence of the series :

(i) $\frac{1^2 \cdot 2^2}{1!} + \frac{2^2 \cdot 3^2}{2!} + \frac{3^2 \cdot 4^2}{3!} + \dots\dots\dots$

(ii) $\sum_{n=1}^{\infty} \frac{1.3.5.....(2n-1)}{2.4.6......(2n)} x^n$. 5,5,5

4. (a) Let $f$ be a function defined on **R** by :

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ -1 & \text{if } x \text{ is irrational} \end{cases}$$

Show that $f$ is discontinuous at every point of **R**.

(b) Let $f$ be a continuous function on $[a, b]$ and $c_1, c_2, \dots\dots$, $c_n$ be points of $[a, b]$. Show that there exists a point $c \in [a, b]$ such that $f(c)$ takes the value average of the values $f(c_1), f(c_2), \dots\dots\dots\dots, f(c_n)$.

(c) Give an example of continuous function which is not uniformly continuous. $4\frac{1}{2}, 4\frac{1}{2}, 4\frac{1}{2}$

5. (a) State and prove Rolle's theorem.

(b) Prove that $|\tan^{-1} x - \tan^{-1} y| \le |x - y|, \quad \forall x, y \in \mathbb{R}$.

(c) Obtain Maclaurin's series expansion of $\sin x$.

$4\frac{1}{2}, 4\frac{1}{2}, 4\frac{1}{2}$

6. (a) Show that the function $f(x) = \sin x(1 + \cos x)$ has a maximum value, when $x = \frac{\pi}{3}$.

(b) Prove that :

$$x - \frac{x^2}{2} < \log(1 + x) < x - \frac{x^2}{2(x+2)} \quad \forall x > 0.$$

(c) Find the value of :

$$\lim_{x \to 0} \left( \frac{1}{x^2} - \frac{1}{\sin^2 x} \right).$$

$4\frac{1}{2}, 4\frac{1}{2}, 4\frac{1}{2}$