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Your Roll No. ....

5123

B.Sc. Prog./III **B**

MA301—MATHEMATICS

(Real Analysis)

(For Physical Sciences/Applied Sciences)

(Admissions of 2008 and onwards)

Time : 3 Hours *Maximum Marks : 112*

(Write your Roll No. on the top immediately on receipt of this question paper.)

All questions are compulsory.

Attempt any *two* parts from each question.

1. (*a*) Define an uncountable set. Show that every subset of a countable set is countable. 2+4=6

P.T.O.

(*b*) Define supremum and infimum of a set. Find supremum of the set $S = \{r \in Q \mid r < a,\ a \in \mathbf{R}\}$. 2+4=6

(*c*) State and prove Archimedean property of real numbers. Show that the infimum of the set : 3+3=6

$$S = \left\{\frac{1}{n+1} \middle| n \in \mathbf{N}\right\} \text{ is '0'.}$$

2. (*a*) Prove that a point (p is a limit point) of a set A if and only if every neighbourhood of p contains infinitely many points of A. 6

(*b*) Prove that every convergent sequence is bounded. Justify by giving an example that the converse is not true. 4+2=6

(*c*) State Bolzano Weierstrass Theorem. Justify that no condition in the theorem can be dropped. 2+4=6

3. (*a*) If $\langle a_n \rangle$ be a sequence such that, $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l$, where $| l | < 1$, then prove that $\lim_{n\to\infty} a_n = 0$. 6

(*b*) Show that the sequence $\langle a_n \rangle$ defined by :

$$a_n = 1 + \frac{1}{4} + \frac{1}{7} + \ldots\ldots\ldots\ldots + \frac{1}{3n-2}$$

is not convergent. 6

(*c*) Show that the sequence $\langle s_n \rangle$ defined by the formula :

$S_{n+1} = \sqrt{3S_n}$, $S_1 = 1$ converges to 3. 6

4. (*a*) Let Σu_n and Σv_n be two positive term series such that :

$$\lim_{n\to\infty} \frac{u_n}{v_n} = l \quad (l \text{ is finite and non-zero})$$

then, prove that Σu_n and Σv_n converge or diverge together. 8

(b) Test for the convergence the series : 3+3+2=8

(i) $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n-1}}{n}$

(ii) $\sum_{n=1}^{\infty} \sin \frac{1}{n^2}$

(iii) $\sum_{n=1}^{\infty} \frac{5^n+5}{3^n+2}$

(c) Define absolute and conditional convergence of a series.

Investigate the convergence of the series : 3+5=8

$$\frac{\sin\sqrt{1}}{1} - \frac{\sin\sqrt{2}}{2^{3/2}} + \frac{\sin\sqrt{3}}{3^{3/2}} + \ldots\ldots\ldots\ldots$$

5. (a) Show that the sequence $\langle f_n \rangle$: where :

$$f_n(x) = \frac{nx}{1+n^2x^2}, x \in \mathbf{R}$$

is point-wise convergent but is not uniformly convergent is any interval containing zero. 4+4=8 (*b*) Show that the series 5+3=8

$$\sum_{n=1}^{\infty} \frac{x}{(nx+1)\{(n-1)x+1\}}$$

is uniformly convergent on any interval $[a, b]$, $0 < a < b$, but only pointwise on $[0, b]$.

(*c*) Let $<f_n>$ be a sequence of function, such that :

$$\lim_{n \to \infty} f_n(x) = f(x), \; x \in [a, b]$$

and let $M_n = \sup_{x \in [a, b]} |f_n(x) - f(x)|$.

Show that $f_n \longrightarrow f$ uniformly on $[a, b]$ if and only if $M_n \to 0$ as $n \to \infty$. 8

6. (*a*) Show that :

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\ldots, \; -1 < x \leq 1$$

and deduce that :

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots\ldots\ldots$$

6+2=8

(*b*) Find the interval of convergence of the series : 8

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (x-1)^n .$$

(*c*) Determine the expansion of $\cos x$ in terms of power series. 8

7. (*a*) Show that the function f defined by :

$$f(x, y) = \frac{x^2 y}{x^4 + y^2} \text{ if } (x, y) \neq (0, 0)$$

and $f(0, 0) = 0$ is not continuous at origin. 7

(*b*) Discuss the following function for continuity and differentiability at origin : 3+4=7

$$f(x, y) = \frac{xy^2}{x^2 + y^2},$$

where $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$.

(c) State Schwarz's theorem. Show that the function :

$$f(x, y) = \frac{xy(x^2 - y^2)}{x^2 + y^2}, \text{ if } (x, y) \neq (0, 0)$$

$$f(0, 0) = 0$$

does not satisfy the condition of Schwarz's theorem and

$$f_{xy}(0, 0) \neq f_{yx}(0, 0).$$ 2+3+2=7

8. (a) Let f be a function defined by :

$$f(x, y) = \frac{x^3 + y^3}{x - y} \quad \text{if } x \neq y$$

$$f(x, y) = 0 \quad \text{if } x = y$$

show that f possesses a directional derivative in every direction at (0, 0) but is not continuous at (0, 0). 4+3=7

(b) Show that for $0 < \theta < 1$.

$$\sin x . \sin y = xy - \frac{1}{6}\left[(x^3 + 3xy^2)\right.$$

$$\left.\cos\theta x . \sin\theta y + (y^3 + 3x^2 y)\sin\theta x . \cos\theta y\right]$$ 7

(c) Show that the function :

$$f(x, y) = 2x^4 - 3x^2 y + y^2$$

has neither maximum nor minimum at (0, 0). 7