[This question paper contains 6 printed pages.]

1928 *Your Roll No.* ....

B.Sc. Prog. / III **E**

MA 301 – MATHEMATICS

(Real Analysis)

(For Physical Sciences / Applied Sciences)

(Admissions of 2008 and onwards)

Time : 3 Hours *Maximum Marks : 112*

(Write your Roll No. on the top immediately on receipt of this question paper.)

All *the questions are compulsory.*
Attempt any **two** *parts from each question.*

1. (a) Define an upper bound and supermum of a bounded set of real numbers. Show that for a non-empty set S, an upper bound M is the supermum if for each $\in > 0$, if a real number $x \in S$ such that $x > M - \in$.

(b) Define limit point of a set. Give an example of each of the following sets.

(i) A set which has no limit point.

(ii) A set which has infinite number of limit points lying inside the set.

(iii) A set which has infinite number of limit points lying outside the set.

(c) Define order axioms and completeness axioms in R. Give an example of a field

(i) which is complete.

(ii) which is not complete.

Justify your answer.

2. (a) Prove that every convergent sequence is bounded. Is the converse true ? When is a bounded sequence convergent ?

(b) State and prove a sufficient condition for convergence of a monotonic increasing sequence.

(c) Let $<S_n>$ be a sequence such that

$$S_{n+1} = 2 - \frac{1}{S_n}, \ n \geq 1$$

$$S_1 = \frac{3}{2}$$

Show that $<S_n>$ is convergent to 1.

3. (a) Show that the sequence $<r^n>$ where $-1 < r \le 1$ is convergent.

(b) State and prove necessary condition for convergence of a series $\sum_{n=1}^{\infty} u_n$. Is the condition sufficient. Justify by giving example.

(c) State Leibnitz test for alternating series. Show that $\frac{1}{1^p} - \frac{1}{2^p} + \frac{1}{3^p}$ .... converges for $p > 0$.

4. (a) State and prove Couchy's Root Test for positive term series.

(b) Test for convergence the series

(i) $\sum_{n=1}^{\infty} \frac{n^2 - 1}{n^2 + 1} x^n, x > 0$

(ii) $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}, x \in R$

(c) Test for convergence the series

(i) $1 + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \ldots\ldots\ldots\ldots$

(ii) $\sum_{n=1}^{\infty}\left(\sqrt{n^4+1}-\sqrt{n^4-1}\right)$

(iii) $1+\frac{1}{2^2}+\frac{1}{3^3}+\frac{1}{4^4}+\ldots\ldots\ldots\ldots$

5. (a) Show that the sequence $<f_n(x)>$ where $f_n(x)=\frac{1}{x+n}$ is uniformly convergent in any interval $[0,b]$, $b>0$.

(b) Show that the series

$$x^4+\frac{x^4}{1+x^4}+\frac{x^4}{\left(1+x^4\right)^2}+\frac{x^4}{\left(1+x^4\right)^3}+\ldots\ldots\ldots\ldots$$

is not uniformly convergent on $[0,1]$.

(c) Show that $<f_n(x)>$ where $f_n(x) = nxe^{-nx^2}$ converges pointwise on $[0,1]$ and $\lim_{n\to\infty}\int_0^1(x)d_x \neq \int_0^1 f\,dx$ where $f(x)$ is pointwise [illegible] of $<f_n(x)>$.

What do you conclude from this inequality ?

6. (a) Find the radius of convergence of the series

$$x+\frac{x^2}{2^2}+\frac{2!x^3}{3^3}+\frac{3!x^3}{4^4}+\ldots\ldots\ldots\ldots$$

(b) Show that $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3}$ ...., $-1 \le x < 1$

and deduce that $\log 2 = 1 - \frac{1}{2} + \frac{1}{3}$ ....

(c) Find power series expansion of sin x.

7. (a) Let $f(x,y) = \begin{cases} y + x\sin\frac{1}{y}, & y \neq 0 \\ 0, & y = 0 \end{cases}$

Show that $\log_{(x,y)\to(0,0)} f(x,y)$ and $\lim_{y\to 0} \lim_{x\to 0} f(x,y)$ exists but $\lim_{x\to 0} \lim_{y\to 0} f(x,y)$ doesnot exists.

(b) Let $f(x,y) = \begin{cases} x\sin\frac{1}{x} + y\sin\frac{1}{y}, & xy \neq 0 \\ x\sin\frac{1}{x}, & x \neq 0 \\ y\sin\frac{1}{y}, & y \neq 0 \\ 0, & x = 0 = y \end{cases}$

Show that f is not differentiable at (0,0).

(c) Let $f(x,y) = \begin{cases} xy(x^2 - y^2) \\ 0 \end{cases}$

Show that $f_{xy}(0,0) \neq f_{yx}(0,0)$.

8. (a) Find the points of maxima and minima for the function

$$f(x,y) = x^2 - 3xy^2 + 2y^4.$$

(b) Expand $x^4 + x^2y^2 - y^4$ about the point (1,1) upto terms of second degree.

(c) Let $f(x,y) = xy$.

Find directional derivative of (–1,2) in the direction $i + 2j$.