[This question paper contains 4 printed pages.]

Sr. No. of Question Paper : 8355 **C** **Roll No............**

Unique Paper Code : 235566

Name of the Paper : MAPT-505 : Mathematics–V Real Analysis

Name of the Course : B.Sc. Physical Science, Part III

Semester : V

Duration : 3 Hours

Maximum Marks : 75

Instructions for the Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. **All** questions are compulsory.
3. *Attempt any* **two** parts from each question.

1. (a) Define infimum of a bounded set. Give an example of each of the following :

   (i) A set having an infimum but not a supremum.

   (ii) A set having neither an infimum nor a supremum.

   Prove that the smallest member of a set, if it exists, is the infimum of the set. (5)

   (b) Define countable set. Prove that if A_m is a countable set for each $m \in \mathbb{N}$, then the union $A = \bigcup_{m=1}^{\infty} A_m$ is also a countable set. (5)

   (c) Prove that every non-empty set of real numbers which is bounded below has an infimum. (5)

2. (a) Show that

   (i) The set $\mathbb{Z}$ of integers has no limit points.

   (ii) Derived set of $]a,b[$ is $[a,b]$. (5)

(b) State and prove Archimedean property of real numbers. Use it to prove that

if $S = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$, then $\inf S = 0$. (5)

(c) If $<a_n>$ be a sequence such that $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = l$, where $|l| < 1$, then prove that

$\lim\limits_{n\to\infty} a_n = 0$. (5)

3. (a) Show that

(i) $\lim\limits_{n\to\infty}\left[\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \ldots + \frac{1}{\sqrt{n^2+n}}\right] = 1$.

(ii) $\lim\limits_{n\to\infty}\left\{\frac{(3n)!}{(n!)^3}\right\}^{1/n} = 27$. (6)

(b) State Cauchy's convergence criterion for sequence. Check whether the sequence $<S_n>$, where $S_n = 1 + \frac{1}{4} + \frac{1}{7} + \ldots + \frac{1}{3n-2}$ is convergent or not. (6)

(c) Show that a monotonically increasing sequence is either convergent to its supremum or diverges to $+\infty$. (6)

4. (a) Prove that a necessary condition for convergence for an infinite series $\sum u_n$ is that $\lim\limits_{n\to\infty} u_n = 0$. Is the converse true ? Justify.

Show that the series $\sum\limits_{n=1}^{\infty}\left(\frac{1}{n}\right)^{1/n}$ is not convergent. (7)

(b) Test the convergence of the series :

(i) $\sum \frac{\sqrt{n+1} - \sqrt{n-1}}{n}$

(ii) $\sum 3^{-n-(-1)^n}$

(iii) $\frac{x}{\sqrt{5}}+\frac{x^3}{\sqrt{7}}+\frac{x^5}{\sqrt{9}}+\ldots \qquad (x>0)$ (2+2+3)

(c) Define absolute and conditional convergence for an infinite series. Test the convergence and absolute convergence of the series :

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^p},\ (p>0).$$ (7)

5. (a) State and Prove Cauchy n[th] root test for infinite series. (7)

(b) Determine the interval of convergence of the power series :

$$\sum_{n=1}^{\infty}\frac{(-1)^n(x-1)^n}{2^n(3n-1)}$$ (7)

(c) Define sine function in terms of power series. Prove that

(i) $S(x-y)=S(x)C(y)-C(x)S(y)$

(ii) $C(x-y)=C(x)C(y)+S(x)S(y) \qquad \forall x,y\in\mathbb{R}$,

where C, S denote cosine and sine respectively. (7)

6. (a) State M_n-Test for uniform convergence of sequence of functions. Show that the sequence $\langle f_n\rangle$, where $f_n(x)=\frac{nx}{1+n^2x^2}$ is not uniformly convergent on [0, 1]. ($7\frac{1}{2}$)

(b) Show that the series $\sum_{n=1}^{\infty}\left(\frac{n}{x+n}-\frac{n-1}{x+n-1}\right)$ is uniformly convergent in $[0,k]$ (where k is a positive real number) but not uniformly convergent in $[0,\infty[$. ($7\frac{1}{2}$)

(c) Prove that the uniform limit of a uniformly convergent sequence of continuous functions is continuous and hence deduce that the sequence $\langle f_n \rangle$ where $f_n(x) = x^n$ is not uniformly convergent on $[0,1]$. $(7\frac{1}{2})$