[This question paper contains 4 printed pages.]

Sr. No. of Question Paper : 1596 **F-3** **Your Roll No...............**

Unique Paper Code : 2351302

Name of the Course : **B.Sc. (H) Math – II**

Name of the Paper : Analysis – II (Real Functions)

Semester : III

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt any **three** parts from each question.
3. **All** questions are compulsory.

1. (a) Using the $\varepsilon - \delta$ definition of limit, establish that

$$\lim_{x \to c} \sqrt{x} = \sqrt{c}$$

for any $c \geq 0$. (5)

(b) Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ and $c \in \mathbb{R}$ be a cluster point of A. Prove that if

$$\lim_{x \to c} f(x) > 0,$$

then there exists a neighbourhood $V_\delta(c)$ of c such that $f(x) > 0$ for all $x \in A \cap V_\delta(c)$, $x \neq c$. (5)

(c) State squeeze theorem. Hence or otherwise, show that

$$\lim_{x \to 0} \frac{\sin x}{x}$$

exists and equals 1. (5)

(d) Let $A \subseteq \mathbb{R}$, $f: A \to \mathbb{R}$ and $c \in \mathbb{R}$ be a cluster point of the sets $A \cap (-\infty, c)$ and $A \cap (c, \infty)$. Define the one-sided limits

$$\lim_{x \to c-} f(x) \text{ and } \lim_{x \to c+} f(x)$$

of f at c. Prove that if both one-sided limits of f at c exist and equal L, then

$$\lim_{x \to c-} f(x)$$

exists and equals L. (2+3)

2. (a) Prove that a function $f: A \to \mathbb{R}$ is continuous at the point $c \in A$ if and only if for every sequence (x_n) in A that converges to c, the sequence $(f(x_n))$ converges to f(c). (5)

(b) Show that the function $f: \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 2x & \text{if } x \text{ is rational} \\ x+3 & \text{if } x \text{ is irrational} \end{cases}$$

is continuous only at $x = 3$. (5)

(c) Prove that a real-valued continuous function f on a closed and bounded interval [a, b] is bounded on [a, b]. Is the function $f(x) = 1/x$ bounded on (0,1) ? Justify your answer. (3+2)

(d) Let $a, b \in \mathbb{R}$ with $a < b$. For a continuous function $f: [a,b] \to [a,b]$, show that there exists a point $x_0 \in [a,b]$ with $f(x_0) = x_0$. (5)

3. (a) Show that if f is uniformly continuous on a set $A \subseteq \mathbb{R}$, then it is continuous on A. Show by means of an example that converse may not hold. Under what conditions, does the converse hold ? (2+2+1)

(b) Show that the function $f(x) = \sin x$ is uniformly continuous on the set $[0, \infty)$ but $g(x) = \sin x^2$ is not uniformly continuous on $[0, \infty)$. (2+3)

(c) Examine the differentiability of the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x & , \text{ if } x < 0 \\ 2x-3 & , \text{ if } 0 \le x < 1 \\ 5x+3 & , \text{ if } x \ge 1 \end{cases}$$

at the points $x = -1$ and $x = 1$. (5)

(d) Let I be an interval in $\mathbb{R}$, $f: I \to \mathbb{R}$ and $c \in I$. Show that if there exists a function $\phi : I \to \mathbb{R}$ that is continuous at c and satisfies

$$f(x) - f(c) = \phi(x)(x - c) \text{ for all } x \in I,$$

then f is differentiable at c and $f'(c) = \phi(c)$. Hence or otherwise, determine the derivative of the function $f(x) = x^{1/3}$ at $x = c$, $c \in (0,\infty)$. (3+2)

4. (a) State mean value theorem. If $f, g: [a, b] \to \mathbb{R}$ are continuous on $[a, b]$, differentiable on (a, b) and $f'(x) - g'(x)$ for all $x \in (a, b)$, then prove that f and g differ by a constant on $[a, b]$. (2+3)

(b) Prove that $e^{x-2} \ge x - 1$, $\forall x \in \mathbb{R}$, with equality occurring if and only if $x = 2$. (5)

(c) State interior extremum theorem. Can the conditions in the hypothesis of the theorem be relaxed ? Justify your answer. (2+3)

(d) State Darboux's theorem. Hence or otherwise, show that the function $f: [2,4] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 2 & \text{if } 2 \le x < 3 \\ -3 & \text{if } 3 \le x \le 4, \end{cases}$$

is not a derivative of any real valued function on [2,4]. (2+3)

5. (a) Show that :

$$1 + \frac{1}{2}x - \frac{1}{8}x^2 \le \sqrt{1+x} \le 1 + \frac{1}{2}x,$$

for all $x > 0$. (5)

P.T.O.

(b) Obtain Maclaurin's series expansion of the function $\log(1 + x)$, for $0 < x \leq 1$. (5)

(c) Use Taylor's theorem to determine an approximate value of e so that the error in the approximation does not exceed 10^{-5}. [Given that 9! = 362,880 and 8! =40320] (5)

(d) Define and explain geometrically a convex function on an interval $I \subseteq \mathbb{R}$. Give one example each of a convex function and a non-convex function on $(-5, 5)$. (1+2+1+1)