[This question paper contains 4 printed pages.]

Sr. No. of Question Paper : 1595 **F-3** **Your Roll No...............**

Unique Paper Code : 2351301

Name of the Course : **B.Sc. (H) Mathematics**

Name of the Paper : Algebra – II (Group Theory I)

Semester : III

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt any **two** parts from each question.
3. **All** questions are compulsory.

1. (a) Let a and b be elements of an Abelian group and let n be any integer. Show that $(ab)^n = a^n b^n$. Is this also true for non-Abelian groups ?

   (b) Let H be a subgroup of a group G. Define the centralizer of H. Show that the centralizer of H is a subgroup of G.

   (c) (i) Let $H = \{a + ib : a, b \in \mathbb{R}, ab \geq 0\}$. Prove or disprove that H is a subgroup of $\mathbb{C}$ under addition.

   (ii) Let G be a group such that $ab = ca \Rightarrow b = c \ \forall\, a, b, c \in G$. Show that G is Abelian. (6,6,6)

2. (a) Let $G = \langle a \rangle$ be a cyclic group of order n. Then, $G = \langle a^k \rangle$ if and only if $\gcd(k, n) = 1$.

   (b) If the pair of cycles $\alpha = (a_1, a_2, \ldots, a_m)$ and $\beta = (b_1, b_2, \ldots, b_n)$ have no entries in common, then show that $\alpha\beta = \beta\alpha$.

(c) (i) Let $H = \{\beta \in S_5 : \beta(1) = 1 \text{ and } \beta(3) = 3\}$. Prove that H is a subgroup of S_5.

(ii) Let $\beta = (1\ 2\ 3)(1\ 4\ 5)$. Write β^{99} in cycle form. (6,6,6)

3. (a) If G is a finite group and H is a subgroup of G. then show that |H| divides |G|. Moreover, prove that the number of distinct left (right) cosets of H in G is $\frac{o(G)}{o(H)}$. Also, show that in a finite group, the order of each element of the group divides the order of the group.

(b) (i) State and prove Orbit-Stabilizer Theorem.

(ii) Let $G = \{(1), (132)(465)(78), (132)(465), (123)(456), (123)(456)(78), (78)\}$. Find $orb_G(1)$, $orb_G(4)$ and $stab_G(7)$.

(c) (i) Let G be the group of non-zero complex numbers under multiplication and let $H = \{x \in G : |x| = 1\}$. Give a geometric description of the cosets of H in G.

(ii) If a group G has no non-trivial subgroup, show that G must be finite of prime order. (6.5,6.5,6.5)

4. (a) (i) If N is a normal subgroup of G and $\left|\frac{G}{N}\right| = m$, show that $x^m \in N, \forall x \in G$.

(ii) Let H be a normal subgroup of a finite group G. If $\gcd\left(|x|, \left|\frac{G}{H}\right|\right) = 1$, show that $x \in H$.

(b) (i) Prove that a subgroup H of G is normal in G if and only if :

$$xHx^{-1} \subseteq H, \quad \forall x \in G.$$

(ii) Let H be a subgroup of a group G such that $x^2 \in H$, $\forall\, x \in G$. Show that H is a normal subgroup of G.

(c) (i) Prove that A_n is normal in S_n.

(ii) Let N be a normal subgroup of a group G. If N is cyclic, prove that every subgroup of N is also normal in G. (6.5,6.5,6.5)

5. (a) If M and N are normal subgroups of a group G and $N \le M$, prove that

$\frac{G/N}{M/N}$ is isomorphic to $\frac{G}{M}$.

(b) Let G be a group of permutations and consider the multiplicative group $\{1,-1\}$. For each $\sigma \in G$, define

$$\Phi : G \to \{1,-1\}$$

by

$$\Phi(\sigma) = \begin{cases} 1, & \text{if } \sigma \text{ is an even permutation} \\ -1, & \text{if } \sigma \text{ is an odd permutation} \end{cases}$$

Prove that Φ is a homomorphism. Also, find Ker Φ.

(c) Prove that there is no onto homomorphism from $Z_8 \oplus Z_2$ to $Z_4 \oplus Z_4$.

(6,6,6)

6. (a) State and prove Cayley's theorem.

(b) Let $\varphi : G \to G'$ be an isomorphism of a group G onto a group G′.

Then, prove that :

(i) G is cyclic if and only if G′ is cyclic.

(ii) For any elements a and b in G, a and b commute if and only if $\varphi(a)$ and $\varphi(b)$ commute.

(iii) If K is a subgroup of G, then $\varphi(K) = \{\varphi(k) : k \in K\}$ is a subgroup of G′.

(c) (i) Prove that $\mathbb{Z}$ under addition is not isomorphic to $\mathbb{Q}$ under addition.

(ii) Let φ be a homomorphism from U(30) to U(30) such that $\varphi(7) = 7$ and Ker $\varphi = \{1,11\}$. Find all x such that $\varphi(x) = 7$. (6.5,6.5,6.5)