[This question paper contains 2 printed pages.]

**Sr. No. of Question Paper : 1116** **G** **Your Roll No...............**

Unique Paper Code : 235304

Name of the Paper : III.3 – Algebra II

Name of the Course : **B.Sc. (Hons.) Mathematics, Part II**

Semester : III

Duration : 3 Hours Maximum Marks : 75

**Instructions for Candidates**

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt any **two** parts from each question.
3. **All** questions are compulsory.

1. (a) Let $G = Q\backslash\{-1\}$. Define * on G by $a*b = a + b + ab$ for all $a,b \in G$. Prove that $(G,*)$ is an abelian group. (6)

   (b) (i) Let G be a group and H be a non-empty subset of G. Prove that H is a subgroup of G iff $a.b^{-1} \in H$ for all $a,b \in H$. (3)

   (ii) Prove that the group of positive rational numbers under multiplication is not cyclic. (3)

   (c) Define centralizer of $a \in G$, where G is a group.

   (i) Prove that C(a) is an subgroup of G.

   (ii) Let G = GL(2, R). Find $C\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. (6)

2. (a) For every non-negative integer n, prove that nZ is a subgroup of (Z,+). Moreover, prove that every subgroup of Z is of the form mZ for some non-negative integer m. (6)

   (b) Let G be a group. Let $a,b \in G$ such that $ab = ba$ and $|a| = m$ and $|b| = n$. If $\langle a\rangle \cap \langle b\rangle = \{e\}$. Prove that G has an element of order l.c.m.(m, n). (6)

   (c) Prove that the order of a cyclic group is equal to the order of its generator. (6)

P.T.O.

3. (a) Let H and K be two subgroup of G. Prove that HK is a subgroup of G iff HK = KH. (6.5)

(b) (i) Show that for $n \leq 3$, $Z(S_n) = \{e\}$. (3.5)

(ii) Suppose G is an abelian group with odd number of elements. Show product of all elements of G is identity. (3)

(c) State Lagrange's Theorem and prove with the help of example that converse of Lagrange's Theorem does not hold in general. (6.5)

4. (a) Prove that every permutation of a finite set can be written as product of disjoint cycles. (6.5)

(b) (i) Let H be a subgroup of G and let $a, b \in G$. Prove that either $aH = bH$ or $aH \cap bH = \phi$. (3)

(ii) Let N be a normal subgroup of a group G. If N is cyclic, prove that every subgroup of N is also normal in G. (3.5)

(c) (i) Given that G is a cyclic group, prove that G/N is also cyclic where N is a subgroup of G. Also give an example to show that converse is not true. (3.5)

(ii) If N is a normal subgroup of G and $|G/N| = m$, prove that $x^m \in N$ for all x in G. (3)

5. (a) Define automorphism and inner automorphism induced by an element 'a' of a group G. Prove that Inn (G) is a normal subgroup of Aut G. (6)

(b) Let $\phi$ be a homomorphism from a group G to a group $\bar{G}$. If $\bar{K}$ is a normal subgroup of $\bar{G}$, then prove that $\phi^{-1}(\bar{K})$ is a normal subgroup of G. (6)

(c) Let G = U(10). Find the left regular representation $\bar{G}$ of G and verify that $G \approx \bar{G}$. (6)

6. (a) If M and N are normal subgroups of G and $N \leq M$, prove that $(G/N)/(M/N) \approx G/M$. (6.5)

(b) (i) Let $\phi: G \to \bar{G}$ be a homomorphism. Define Ker $\phi$ and prove that Ker $\phi$ is a normal subgroup of G. (3)

(ii) Show $\phi: Z_{12} \to Z_{12}$ by $\phi(x) = 3x$ is a homomorphism and find Ker $\phi$. Also find $\phi^{-1}(6)$. (3.5)

(c) Let k be a divisor of n. Prove that $Z_n/\langle k\rangle \approx Z_k$. (6.5)