This question paper contains **4+1** printed pages]

Roll No. | | | | | | | | | | | |

S. No. of Question Paper : **7850**

Unique Page Code : **2351201** **F-2**

Name of the Paper : **Analysis-I (Real Analysis) [DC-1.3]**

Name of the Course : **Bachelor with Honours in Mathematics**

Semester : **II**

Duration : **3 Hours** Maximum Marks : **75**

(Write your Roll No. on the top immediately on receipt of this question paper.)

Attempt any *two* parts from each question.

All questions are compulsory.

1. (*a*) If A is a set with m elements and B is a set with n elements and if $A \cap B = \phi$ then show that $A \cup B$ has $m + n$ elements. 5

(*b*) Determine the set :

$$A = \left\{ x \in \mathbf{R} : \frac{2x+1}{x+2} < 1 \right\}$$ 5

(*c*) Let $a \in R$ show that the intersection and the union of any two neighbourhoods of a are again neighbourhoods of a. 5

P.T.O.

2. (a) Show that there exist a positive real number x such that $x^2 = 2$. 5

(b) Show that :

$$\text{Sup}\left\{1 - \frac{1}{n} : n \in \mathbf{N}\right\} = 1.$$ 5

(c) Define limit point of a set. Find limit points of [0, 1]. 5

3. (a) (i) Define the convergence of sequence (x_n) of real numbers and show that limit of a sequence is unique. 2.5

(ii) Show that :

$$\lim (n)^{\frac{1}{n}} = 1$$

and hence find :

$$\lim \left(\sqrt{n}\right)^{\frac{1}{2n}}.$$ 5

(b) (i) Show that :

$$\lim \frac{2^n}{n!} = 0.$$ 2.5

(ii) Suppose (x_n) is a sequence of positive real numbers such that :

$$\lim \frac{x_{n+1}}{x_n} = l.$$

If $l < 1$, then show that (x_n) converges and $\lim x_n = 0$. 5

(*c*) (*i*) State Monotone Convergence Theorem for Sequences. 2.5

(*ii*) Show that the sequence (x_n), where $x_1 \geq 2$ and

$$x_{n+1} = 1 + \sqrt{x_{n-1}}$$

for $n \in \mathbf{N}$ is a decreasing sequence bounded below by 2. Find the limit. 5

4. (*a*) (*i*) Determine the limit of the sequence :

$$\left(1 + \frac{1}{2n}\right)^n$$ 2.5

(*ii*) Suppose that $x_n \geq 0$ for all $n \in \mathbf{N}$ and

$$\lim_{n \to \infty} \left\{(-1)^n x_n\right\}$$

exists. Show that the sequence (x_n) converges. 5

(*b*) (*i*) State Bolzano-Weierstrass Theorem for sequences. Give an example of an unbounded sequence that has a convergent subsequence. 2.5

(*ii*) If $x_1 < x_2$ are arbitrary real numbers and

$$x_n = \frac{1}{2}\left(x_{n-1} + x_{n-2}\right) \text{ for } n > 2.$$

Show that the sequence (x_n) is convergent. Find its limit. 5

(c) (i) Show that the sequence :

$$1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots\ldots + \frac{1}{n!}$$

is a Cauchy sequence. 2.5

(ii) Show that every Cauchy sequence is bounded. Is the converse true ? Justify your answer. 5

5. (a) Define the convergence of the series :

$$\sum_{n=1}^{\infty} x_n.$$

Show that the series :

$$\sum_{k=1}^{\infty} \left(\frac{k^2}{e^k} - \frac{(k+1)^2}{e^{(k+1)}} \right)$$

converges and its sum is equal to $\frac{1}{e}$. 6

(b) Examine for convergence the series :

(i) $\sum_{n=1}^{\infty} ne^{-n}$

(ii) $\sum_{n=1}^{\infty} \frac{n^2}{(n+1)!}$. 6

(*c*) Show that the series :

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p} \, p \geq 0$$

converges if and only if $p > 1$. 6

6. (*a*) Show that every absolutely convergent series is convergent. Is the converse true ? Justify your answer. 6.5

(*b*) Test the convergence of the series :

(*i*) $\sum_{n=1}^{\infty} \frac{(-1)^n - (-1)^{n+1}}{n+1}$

(*ii*) $\sum_{n=2}^{\infty} \frac{1}{(\log n)^n}$. 6.5

(*c*) State the ratio test for the series of real numbers giving examples where the test fails. Further examine for convergence the series :

$$\sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n + 1}.$$ 6.5